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Using Legendre polynomials Formulas at Fresnel Integral Diffraction

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ABSTRACT:
Many advanced scientific phenomena of optics, light, and physics can be represented mathematically in the form of Volterra integral equations. In this paper, we studied the diffraction phenomena of the light beam. Since the most important category of scalar diffraction theories is the Fresnel diffraction integral. These integrals have been used in numerous studies on the propagation effects of structured light beams. These scalar diffraction theories have been widely utilized in studying the propagation of structured light. finally, we applied first-kind shifted Legendre polynomials to find the interpolate solutions of weakly singular Volterra integral equations of the second kind, where the Fresnel integral of diffraction will be involved. Numerical examples have been included in order to show the efficiency of the presented method. The exact solution of the represented example is compared to the approximate solution and the absolute error is calculated to illustrate the efficiency of the proposed method.

Keywords: Volterra Integral equations; Fresnel diffraction integral; Diffraction theory; Legendre polynomials.

1. Introduction

Francesco Maria Grimaldi, an Italian, was the first to record the phenomena of diffraction in 1665. The waveform at the boundary edge of any wave, including light, gets warped when it passes through a barrier. There will be more noticeable distortion if the wave goes through a gap. The distortion is accentuated when the gap width gets closer to the wave's wavelength. We call this process diffraction. The diffracted light will interfere with one another to form a unique pattern (called the diffraction pattern). Whatever the gap that diffracts the initial light wave is, the diffraction pattern's characteristics will vary. A function that represents diffraction is an integral function known as Fresnel diffraction integral. Since many integral equations that describe physical procedures and phenomena have solutions that are typically represented in terms of special functions. These functions, whose notation has been standardized over time, combine a wide range of difficult mathematical expressions, such as integrals and extremely transcendental functions. These integrals, which are also known as Fresnel integrals, are two transcendental functions that are named for Augustin Jean Fresnel. [1,.6]

The most important category of scalar diffraction theories is the Fresnel diffraction integral, which has been used in numerous studies on the propagation effects of structured light beams. These scalar diffraction theories have been widely utilized in studying the propagation of structured light beams, such as the Hermite–Gaussian beams, Laguerre–Gaussian beams, Bessel beams, and Airy beams. The Fresnel diffraction integral, which approximates the Kirchhoff diffraction formula, is typically used to study the propagation of paraxial light beams in a homogeneous medium. Nevertheless, most of the earlier research only looked at one kind of structured light beam's propagation using a scalar diffraction theory. The studies discussed in [2,3] showed that an accurate method for representing the finite superposition of Gaussian beams, which are then propagated by means of the Fresnel diffraction integral, could be used to calculate the propagating rate for truncated, nondiffracting, and accelerating beams.

The Iterative Fresnel Integrals Method (IFIM) is a computer-based simulation technique that we clarify in this paper. It involves repeatedly calculating Fresnel integrals to obtain the full near-field Fresnel diffraction patterns or images from N-apertures in any given experimental configuration. With this IFIM approach, the images detected in the far-field can be regarded as a particular case. This Fresnel simulation can be run on any computer by using MATLAB programs. In addition to serving as a helpful educational tool for comprehending the specifics of the diffraction procedure, the IFIM approach replicates an actual diffraction experiment on a computer [4]. Linear FM Pulse Compression Spectra were derived by applying Fresnel integrals. There have been approaches for evaluating these integrals for a few years now. Fresnel integrals are assessed using polynomial approximations. In diffraction analysis, these approximations are utilized. [5] provides a methodology to assess Fresnel integrals using the approach of continued fractions. However, Using expanding in terms of Bessel functions is the solution to estimate Fresnel integrals for high values of $|x|$ [7, 8]. Additionally, in [9], the Fresnel integrals asymptotic expansion's first term was supplemented by an exponentially expanding function that reaches infinity at the zero of the Fresnel function argument to provide one of the conducted approximations [6]. Similarly, a number of numerical methods have been put forth to assess Fresnel integrals. for instance, the Fresnel integrals' Chebyshev estimations, that depend on the argument's magnitude, use various argument values at various evaluation intervals. A technique for spreadsheet calculations of Fresnel integrals to six important figures that is based on iterative refinements of existing related estimates that are exact for only three figures.

Most of the recent applications are concerned with integral equations, ordinary differential equations, and partial differential equations. Doan Thi Hong Hai and Nguyen Minh Phu made a Critical Review of three Mathematical Models concerned with ordinary and partial differential equations for Solar

Air Heater Analysis. As a result, researchers tried to find marvelous techniques for solving various kinds of equations [20]. One of the common equivalent boundary integral equations is the weakly singular Fredholm integral equations of the second kind. These equations appear in many engineering fields, such as radiation, potential theory, scattering, electromagnetism, and other scientific fields [21,22]. On the other side the weakly singular Volterra integral equations are examples of evolution issues that appear in a variety of applications, including demography, viscoelastic materials, electromagnetic scattering, and diffraction. The solution of Volterra integral equations with weakly singular kernels has been the subject of numerous articles detailing modern methods and approaches. This work solves the Volterra integral equations, which describe the previous Fresnel integral of the diffraction phenomenon, using first kind shifted Legendre polynomials.

This is how the paper is organized. Section 2 introduces the concept of Fresnel integrals. In Section 3, The Fresnel integrals in diffraction is presented. Solving diffraction problems using Legendre polynomials is performed in Section 4. Examples are introduced in Section 5. Finally, conclusions of this work are provided in Section 6.

2. Fresnel integrals

The Fresnel integrals Fresnel $S(x)$ and Fresnel $C(x)$, denoted by $S(x)$ and $C(x)$ respectively, are two transcendental functions used in optics. They arise in diffraction phenomena. It has the following integrals as its definition:

$$S(p)=\int_0^p \cos\left(\frac{\pi x^2}{2}\right)dx \tag{1}$$

$$C(p)=\int_0^p \sin\left(\frac{\pi x^2}{2}\right)dx \tag{2}$$

where the normalizing factor is denoted by the coefficient $\pi/2$.

3. The Fresnel integrals in diffraction

As a real variable x and a real number $P \geq 0$ respectively. Whenever P goes to infinity, $C = S = 1/2$ [11]. Odd functions of x are the $S(x)$ and $C(x)$ Fresnel integrals respectively. The domain of complex numbers can be included in them; As a result, analytic functions for a single variable can be derived [12].

$$C(x)=\frac{\sqrt{\pi}}{4}\left(\sqrt{i}\,erf\left(\sqrt{i}\,x\right)+\sqrt{-i}\,erf\left(\sqrt{-i}\,x\right)\right) \tag{3}$$

$$S(x)=\frac{\sqrt{\pi}}{4}\left(\sqrt{-i}\,erf\left(\sqrt{i}\,x\right)+\sqrt{i}\,erf\left(\sqrt{-i}\,x\right)\right) \tag{4}$$

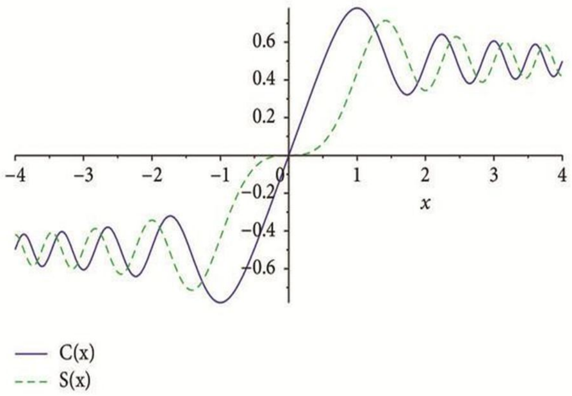

Figure 1: Fresnel integrals

The parametric curve produced by the Fresnel integrals $S(x)$ and $C(x)$ at Eq.(1) ,(2) is called the Cornu-spiral, sometimes called as the clothoid-spiral or Euler-spiral. At the origin, the spiral is a tangential curve to the x-axis, and the distance travelled across it causes its radius of curvature to decrease inversely (Figure 2).

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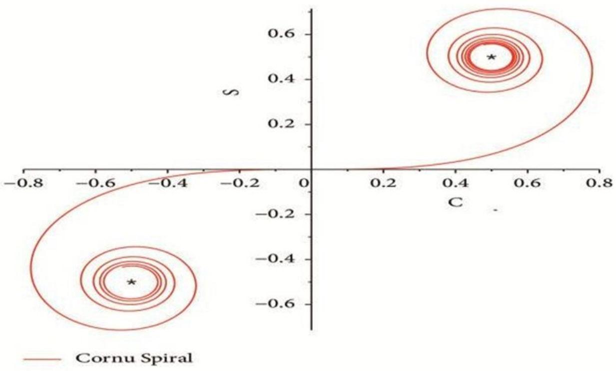

Figure 2: Cornu spiral.

A representation obtained quantitatively from the pattern of diffraction can be obtained in the field of optics by using the Cornu spiral. Accordingly, the diffraction is known as near-field diffraction or Fresnel zone when it is detected from a location near the diffractor obstacle. As such, the semi-periodic Fresnel zones are computed from the observation point for the purpose of conducting diffraction pattern analysis. It follows that each zone's contribution is a cumulative phasor. We find the Cornu spiral at the boundary of the zones of infinitesimal width. The curvature of the Cornu spiral [13] is correlated with the separation between the origin and the curvature at any given location. Because a moving object with this displacement will have a fixed angular acceleration at a constant velocity, this characteristic makes it possible to use for railroad and highway trace. Since the transitional curve has an infinite radius at the point of tangent to the straight portion of the line and a radius R at the point of tangent to the regular circular curve, straight stretch-clothoid-circular-clothoid-straight stretch curves are the most common kind of curves on highways[14, 15].

Finally, this diffraction phenomena appears in the form of Volterra integral equation, which has the following form:

$$z(x) = g(x) + \int_0^x (x-t)^{-\beta} \vartheta(x,t) z(t)\,dt \,, \quad (5)$$

$$t \in \Omega = [0,t],\ \beta \in]0,1[$$

4. Solving diffraction problem using Legendre polynomials

We are now going to start the process of implementing the solution to Equation. (5). The proposed method would use the first kind shifting Legendre polynomials to approximate the known function as well as the unknown function. There will be two approximations of the kernel for its two variables.

Definition A: It is possible to find the set of orthogonal on $[0,1]$ shifted Legendre polynomials of the first kind $\{P_k(x)\}_0^n$ by:-

$$p_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^2 - x)^n \,; \quad (6)$$

$$\int_0^1 p_n(x) p_m(x)\,dx = \frac{1}{2n+}\delta_{mn},\ n = \overline{0:m}$$

Assume that $z(x)$ is individually continuous and that the series $\sum_{k=0}^{\infty} c_k p_k(x)$, where

$$c_k = (2k+1)\int_0^1 z(x) p_k(x)\,dx$$ converges to $z(x)$

if and only if x is not a point of discontinuity, has a finite number of maxima and minima.

Definition B: The Legendre coefficients matrix, indicated by $p_{n,n}(x)$, is the square matrix obtained by extracting the coefficients of Legendre polynomials $\{p_k(x)\}_0^n$ such that the coefficients of $p_0(x)$ in ascending power of x are in the first row, the coefficients of $p_1(x)$ are in the second row, and further on. We find the approximate unknown function of degree n based on A and B, denoted by $z_n(x)$ in the form,

$$z_n(x) = Y(x) P^T_{n,n} Z \tag{7}$$

There, $Y(x) = \left[y^k\right]_{k=0}^{n}$ is a row matrix of the monomial basis functions, $Z = \left[z_k\right]_{k=0}^{n}$ is the unknown coefficients column matrix that needs to be identified, and $P_{n,n}$ can be computed using definition B. In a similar vein, the following form can be used to approximate the provided data function.

$$g_n(x) = Y(x) P^T_{n,n} G \tag{8}$$

Where

$$g_k = (2k+1)\int_0^1 g(x) P_k(x)dx \;,\; k = \overline{0,n} \tag{9}$$

The two variables x and t will be taken into account while approximating the kennel $l(x,t) = \dfrac{1}{(x-t)^{\beta}}$ in the same manner as $z_n(x)$.

When $l(x,t)$ is approximated according to x, $l_n(x,t)$ is obtained via the $(n+1)\times 1$ column matrix $N(t)$ in the following form:

$$l_n(x,t) = Y(x) P^T_{n,n} N(t); N(t) = [n_k(t)]_{k=0}^{n},$$
$$n_k(t) = (2k+1)\int_0^1 l(x,t) P_k(x)dx \tag{10}$$

In addition, every entry $n_i(t)$ $\forall i = \overline{0,n}$ will be estimated in relation to the parameter t, allowing us to obtain $l_{n,n}(x,t)$ through the $(n+1)\times(n+1)$ square knowing kernel's coefficients matrix, meaning $K_{n,n}$ in the form:

$$l_{n,n}(x,t) = Y(x) P^T_{n,n} L_{n,n} P_{n,n} Y^T(t)$$
$$L_{n,n} = \left[l_{ij}\right]_{i,j=0}^{n}, l_{ij} = (2i+1)\int_0^1 n_i(t) P_j(t)dt \tag{11}$$

And we get

$$l_{n,n}(x,t) z_n(t) = Y(x) P^T_{n,n} L_{n,n} P_{n,n} \tilde{Y}(t) P^T_{n,n} Z;$$
$$\tilde{Y}(t) = Y^T(t) Y(t) \tag{12}$$

Substituting $l_{n,n}(x,t) z_n(t)$ of Eq. (12) into Eq. (5), we get

$$z_n(x) = g(x) + Y(x) P^T_{n,n} L_{n,n} P_{n,n} \tilde{\tilde{Y}}(x) P^T_{n,n} Z;$$
$$\tilde{\tilde{Y}}(x) = \int_0^x \tilde{Y}(t)dt \tag{13}$$

$$Y(x) P^T_{n,n} L_{n,n} P_{n,n} \tilde{\tilde{Y}}(x) P^T_{n,n} Z$$
$$= -Y(x) P^T_{n,n} L_{n,n} P_{n,n} \tilde{\tilde{Y}}(x) P^T_{n,n} L_{n,n} P_{n,n} \tilde{\tilde{Y}}(x) P^T_{n,n} Z$$
$$= Y(x) P^T_{n,n} L_{n,n} P_{n,n} \tilde{\tilde{Y}}(x) P^T_{n,n} G \tag{14}$$

As a result, we obtain the unknown coefficients matrix Z through.

$$Z = \left(I_n - L_{n,n} P_{n,n} \tilde{\tilde{Y}}(x) P^T_{n,n}\right)^{-1} G \tag{15}$$

Ultimately, $z_n(x)$ is the approximate answer that we discover by

$$z_n(x) = Y(x) P^T_{n,n} \left(I_n - L_{n,n} P_{n,n} \tilde{\tilde{Y}}(x) P^T_{n,n}\right)^{-1} G \tag{16}$$

5. Numerical Examples

To illustrate the effectiveness of the suggested approach, we will examine a numerical example in this section that includes the Volterra integral equation and considers the Fresnel integrals of the diffraction issue that correspond to Equation (5). MATLAB2019a was used to conduct calculations related to the experiments that were previously stated. This problem was resolved for $n=2,5$ and $x=0.1, 0.2, 0.4, 0.6$. $z(x_i)$ represents the exact solution at $x_i=0.0{:}x/10{:}1.0$, $z_n^x(x_i)$ represents the approximate solution polynomial of degree n for $x=0.1, 0.2, 0.4, 0.6$, and $\Re_n^x(x_i)=\left|z(x_i)-z_n^x(x_i)\right|$ represents the absolute error. Figure (3) related to example 1 to show the graphs of the exact solution and the approximate solution.

Example 1: Consider the problem.

$$z(x)=1-x+\cos x^2-\sqrt{\frac{\pi}{2}}\,C\left(\sqrt{\frac{2}{\pi}}\,x\right)+\int_0^x z(t)dt \quad (17)$$

whose exact solution $z(x)=1+\cos x^2$.

x_i	$z(x_i)$	$z_2^{0.1}(x_i)$	$z_5^{0.1}(x_i)$	$\Re_2^{0.1}(x_i)$	$\Re_5^{0.1}(x_i)$
0	2	1.99998766598	1.99999999897	0.00001233402	0.00000000103
0.1	1.99500416527	1.99500434257	1.99500416499	0.0000001773	0.00000000028
0.2	1.99800665778	1.99800645678	1.99800665751	0.000000201	0.00000000027
0.3	1.95533648912	1.95533567432	1.95533648887	0.0000008148	0.00000000025
0.4	1.92106099400	1.92106087694	1.92106099367	0.00000011706	0.00000000033
0.5	1.87758256189	1.87758213456	1.87758256120	0.00000042733	0.00000000069
0.6	1.82533556149	1.82533534321	1.82533556099	0.00000021828	0.0000000005
0.7	1.76484218728	1.76484132456	1.76484218678	0.00000086272	0.0000000005
0.8	1.69670670934	1.69670667893	1.69670670898	0.00000003041	0.00000000036
0.9	1.62160996827	1.62160899765	1.62160996779	0.00000097062	0.00000000048
1	1.54030230586	1.54030228975	1.54030230478	0.00000001611	0.00000000108

Table 1: $z(x_i)$, $z_n^{0.1}(x_i)$, and $\Re_n^{0.1}(x_i)$ of Example 1 for $n=2,5$

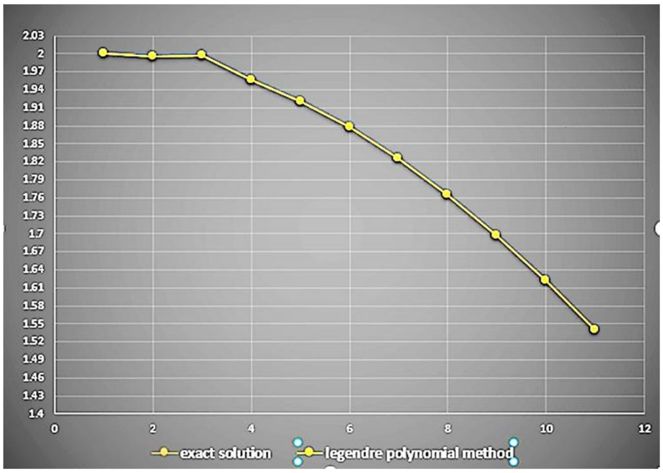

Figure (3). The exact solution and the approximate solution

6. Conclusion

In this study, we used conventional Legendre polynomials to apply numerical solutions to second-kind Volterra integral equations with weakly singular kernels at the Fresnel diffraction integral issues. Operational matrices provide the foundation for this process. An analysis of numerical data was conducted to validate the effectiveness of the suggested approach in diffraction phenomena.

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