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Analysis of Contact Mechanics for Flat Stamps on Graded Coatings Through the application of the First-kind Legendre polynomials in their shifted form

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ABSTRACT:
This study investigates the contact mechanics of graded coatings, focusing specifically on the early fracture behavior of these materials under rigid flat stamp sliding contact loading. The problem is formulated as a weakly singular Fredholm integral equation of the second kind, in this paper we will utilize First-kind Legendre polynomials in their shifted form in a Matrix and vector operation format to approximate this integral. a singularity in the kernel is addressed analytically to ease the analysis [1]. The main goal of this study is to obtain analytical benchmark solutions that enable us to investigate how various characteristics, including the coefficient of friction, material inhomogeneity constants, and characteristic length parameters, affect the critical stresses that could The coatings' fatigue and fracture behavior procedures so The study of integral equations forms a fundamental foundation for solving initial, boundary, and mixed value problems, as it allows for the transformation of any of these problems into a boundary integral equation.

Keywords: Mechanics of contact for graded coatings using a flat stamp ;Fredholm integral equations; Legendre polynomials.

Introduction

Weakly singular integral equations are significantly utilized in diverse fields of science and engineering and play an important role in a variety of amazing domains, including energy harvesting, magnetic disk drives, automotive, aerospace, transportation, micro-electromechanical systems, calculating conformal mappings of domains, exploring electrostatics and low-frequency electromagnetics, analyzing the propagation of acoustic and elastic waves, reformulating radiative heat transfer problems, and investigating hydrodynamic interactions, among elements of a polymer chain, the distribution of surface water waves by a vertical barrier with a gap, and so on. The rigid punch interaction on an elastic half-space was studied by Bakirtas in 1980. By tackling a novel problem where the punch is influenced by a focused normal force, which is possible to mathematically represented as the second kind integral equations with weakly singular kernel, İsa Çömez made a big breakthrough in 2015. This research is built on previous studies [1],[2],[3] The field of integral equations plays a crucial role in solving initial, boundary, and mixed value problems, as it can transform these problems into equivalent boundary integral equations. This transformation helps reduce the computational effort needed for solving boundary problems using traditional methods. Additionally, it allows for handling singularities in the solution that may arise in certain areas of the integration domain. Integral equations have been widely applied across various scientific fields, including artificial intelligence, heat transfer, radar theory, nanotechnology, scattering, and the study of contact mechanics in flat stamp graded coatings, we'll provide insight to Multiphase composites with continuously fluctuating volume fractions that impact their thermomechanical properties are referred to as graded materials, or functionally graded materials (FGMs). In many present and future applications of FGMs, contact concerns—weight transfer challenges in the presence of friction—are common. Gears, cams, machine tools, bearings, and abradable seals are a few examples of structural components used in gas turbines (Trumble et al., 2000; Pan et al., 2003; Miyamoto et al., 1999). An important consideration in the design of load transmission components is surface preparation to reduce the likelihood of cracking. For the best possible design of these components, the necessary material toughness and wear resistance at and at the surfaces are crucial [1], [2],[3] The interaction problem of a functionally graded layer supported by a Winkler foundation and loaded by a rigid cylindrical punch was analyzed by Isa Gomez [5] using linear elasticity theory. In 2015, Gomez made a significant advancement [6] by considering a novel scenario in which the punch moves at a constant subsonic speed and is subjected to a concentrated normal force. By applying the Fourier transform, the governing equations were reduced to a Cauchy singular integral equation, which could be solved using Integration techniques based on Gauss-Chebyshev quadrature.

The way an elastic half-space is affected by a forceful punch was also analyzed by I. Bakirtas [9] in 1980, with the solution reduced to a singular integral equation using the Fourier transform technique. In a separate study, M. A. Guler, and I. Gomez [10] examined the plane contact problem for a rigid cylindrical punch interacting with a functionally graded bilayer. Likewise, this issue was simplified to a single integral equation, with the contact width and pressure determined numerically using the Gauss–Jacobi formula. Finally, the effects of contact stress in frictional contact mechanics involving graded half-planes were explored in [11],[12].

The components of stress intensity and contact stress for the circular stamp profile are calculated numerically and expressed as a second-kind Fredholm integral equation. This study will utilize Through the application of the First-kind Legendre polynomials in their shifted from method to investigate the initiation of fracture in graded coatings under sliding contact stress from a rigid flat stamp [1],[3]

In the following we consider the outline of the paper. In Section 2, the problem formulation of the FGM approximation is presented. In Section 3, We give a few examples to illustrate our point and obtaining the suggested technique's error analysis

Problem formulation

Assume an elastic half-plane for the problem with a functionally graded material (FGM) coating, as illustrated in Fig. (1), which is modeled by a standard stress boundary value problem. The structure consists of a metallic substrate, whose thermo-mechanical properties vary continuously, attached to a metal-ceramic composite covering. The shear modulus of the coating is denoted as $\mu(y)$, as shown in the diagram."

$$\mu(y) = \mu_1 e^{\gamma y}, -A \prec y \prec 0 \quad \textbf{(1)}$$

when the substrate in media 2 is uniform , Let A represent The coating's thickness in (graded 1)., γ is a fixed associated with A material's inhomogeneous structure provided by Eq. (3), and μ_2 is the substrate's Shear modulus that remains constant in the following form

$$\mu_2 = \mu_1 e^{-\gamma y}, -A \prec y \prec 0 \quad \textbf{(2)}$$

$$\gamma = \frac{1}{h} \log \Gamma_3, \Gamma_3 = \frac{\mu_2}{\mu_1} \quad \textbf{(3)}$$

Where $\mu_2 = \mu(y)$ and Poisson's ratio is thought to be insignificant at first glance

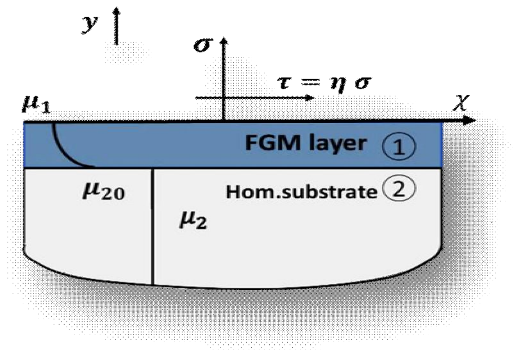

Figure 1: The geometry of the problem involves a homogeneous half-plane coated with a functionally graded material (FGM).

Let us now examine the region $-A \prec y \prec 0$. The issue with Two-dimensional contact could show up as

$$\sigma_{1xx}(x,y) = \frac{\mu(y)}{k-1}\left[(k+1)\frac{\partial u_1}{\partial x} + (3-k)\frac{\partial v_1}{\partial y}\right] \quad (4)$$

$$\sigma_{1yy}(x,y) = \frac{\mu(y)}{k-1}\left[(3-k)\frac{\partial u_1}{\partial x} + (k+1)\frac{\partial v_1}{\partial y}\right] \quad (5)$$

$$\sigma_{1xy}(x,y) = \mu_1(y)\left[\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right] \quad (6)$$

Where $k = (3-m)/(1+m)$ Regarding generalized plane stress, $k = (3-m)$ For the plane strain scenario.

In medium 2, $\mu(y)$ is changed out for μ_2 :

$$(k+1)\frac{\partial^2 m_1}{\partial y^2} + (k-1)\frac{\partial^2 m_1}{\partial x^2} + 2\frac{\partial^2 u_1}{\partial x \partial y} + \gamma(3-k)\frac{\partial u_1}{\partial x} + \gamma(k+1)\frac{\partial m_1}{\partial y} = 0 \quad (7)$$

$$(k+1)\frac{\partial^2 u_1}{\partial x^2} + (k-1)\frac{\partial^2 u_1}{\partial y^2} + 2\frac{\partial^2 m_1}{\partial x \partial y} + \gamma(k-1)\frac{\partial u_1}{\partial y} + \gamma(k-1)\frac{\partial m_1}{\partial x} = 0 \quad (8)$$

$$(k+1)\frac{\partial^2 u_2}{\partial x^2} + (k-1)\frac{\partial^2 u_2}{\partial y^2} + 2\frac{\partial^2 m_2}{\partial x \partial y} = 0 \quad (9)$$

$$(k+1)\frac{\partial^2 m_2}{\partial y^2} + (k-1)\frac{\partial^2 m_2}{\partial x^2} + 2\frac{\partial^2 u_2}{\partial x \partial y} = 0 \quad (10)$$

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Fourier transforms can be used to calculate the displacement components in detail if a rigid, flat stamp with mixed boundary conditions is used. In the region of contact $a \prec x \prec b$, The components of displacement are supplied, while the tractions σ and τ are zero outside this region. The profile of the stamp is shown in Fig. (2)

$$m_1(x,0) = -m_0, \frac{\partial}{\partial x} m_1(x,0) = 0 \qquad (11)$$

The surface displacement is thoroughly discussed in [3], with δ\deltaδ being a specified constant. Consequently

$$-\omega\tau(x) + \frac{1}{\pi}\int_{-a}^{b}\left[\frac{1}{t-x} - k_{11}(t,x)\right]\sigma(t)dt - \frac{1}{\pi}\int_{-a}^{b}\left[\tau(t)k_{12}(t,x)\right]dt = u(x) \qquad (12)$$

$$\omega\sigma(x) + \frac{1}{\pi}\int_{-a}^{b}\left[\frac{1}{t-x} - k_{21}(t,x)\right]\tau(t)dt - \frac{1}{\pi}\int_{-a}^{b}\left[\sigma(t)k_{22}(t,x)\right]dt = n(x) \qquad (13)$$

where $-a \prec x \prec b$, $u(x) = \lambda\frac{\partial}{\partial x}m_1(x,0), n(x) = \lambda u_1(x,0), \lambda = \frac{4\mu_1}{k+1}, \omega = \frac{k-1}{k+1}$

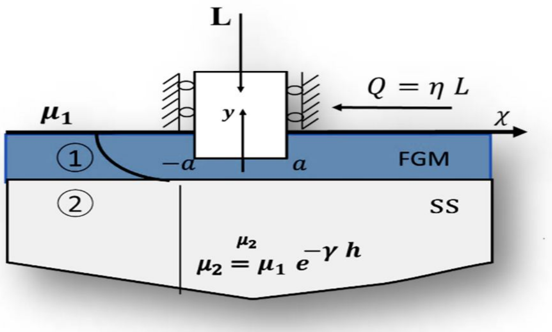

Fig 2: The spatial configuration of the flat stamp problem

a friction in a contact zone follows Coulomb's law, with a constant coefficient of friction η, anytime the stamp shifts in relation to the substrate. This leads to

$$\sigma_{1yy}(x,0) = \sigma(x) = -L(x), \sigma_{1yy}(x,0) = \tau(x) = -\eta L(x) \qquad (14)$$

In which L(X) stands for the pressure, which is unknown, across the contact area, this can be obtained by resolving the integral formula that follows

$$\omega\eta L(x) = \frac{1}{\pi}\int_{-a}^{b}[-\frac{1}{t-k} + k_{11}(t,x) + \eta k_{12}(t,x)]L(t)dt = u(x) \qquad (15)$$

That can be transformed into the form of

$$L(x) + \int_{-a}^{b}\tilde{k}(x,t)L(t)dt = \tilde{u}(x) \qquad (16)$$

In which

$$\tilde{u}(x) = \frac{u(x)}{\omega\eta}, \tilde{k}(x,t) = \frac{\left[\frac{1}{t-x} - k_{11}(t,x) - \eta k_{12}(t,x)\right]}{\omega\eta\pi} \qquad (17)$$

which can be calculated by employing the Shifted Legendre Polynomials of the First Kind [1].

Examine the following: Part II of the Fredholm integral formula:

$$\lambda\tilde{u}(x) = L(x) + \int_0^1 Z(x,t)|x-t|^{-\alpha} L(t)dt \ ; \ 0 \le x \le 1 \ , \ 0 \le \alpha < 1 \qquad (18)$$

Where $\lambda, L(x)$ and the kernel $\tilde{k}(x,t) = Z(x,t)|x-t|^{-\alpha}$ are given, the unknown function $\tilde{u}(x)$ is be determined. The kernel $\tilde{k}(x,t)$ is defined on the square $\{(x,t): 0 \le x, t \le 1\}$. For $\alpha = 0$, (18) his referred to as a non-singular Fredholm integral equation of the second kind. For $0 < \alpha < 1$ (19). No consider Legendre polynomials in their shifted form, which own the property of orthogonality.

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left(x^2 - x\right)^n \ ; \int_0^1 P_n(x)P_l(x)dx = \frac{1}{2n+1}\delta_{ln} \ , \ n = \overline{0:l} \qquad (19)$$

The function $\tilde{u}(x)$ isapproximated based on $\{P_n(x)\}_0^l$,it is obtained in the form

$$\tilde{u}_n(x) = \sum_{i=0}^{n} a_i P_i(x) = \mathrm{X}(x) V \mathrm{A} \quad \textbf{(20)}$$

Here, the known coefficient matrix V is obtained by extracting the coefficients of all the polynomial $P_i(x)\ \forall\ i = \overline{0,n}$ and fulfilled the rows of V. Here $\mathrm{A}=[a_i]_{i=0}^{n}$ These represent the unknown coefficient column matrix that needs to be determined., and $\mathrm{X}(x) = \left[x^i\right]_{i=0}^{n}$ The monomial basis functions' row vector is this one. Likewise, the given data function can be roughly represented as follows

$$L_n(x) = \sum_{i=0}^{n} b_i P_i(x) = \mathrm{X}(x) V L \quad \textbf{(21)}$$

where $\mathrm{L}=[L_i]_{i=0}^{n}$ This stand for the column matrix of known coefficients. such that $[L_i]_{i=0}^{n}$ can be found by

$$L_i = (2\mathrm{i}+1)\int_0^1 L(x) P_i(x)\, dx \quad \textbf{(22)}$$

A kernel $\tilde{k}(x,t)$ It will be roughly calculated in the same way, along with. $\tilde{u}_n(x)$ However, the two variables x and t will be considered when doing this. "Approximating $\tilde{k}(x,t)$ subjected to x , gives $\tilde{k}_n(x,t)$ via the $(n+1)\times 1$ column matrix $\mathrm{N}(t)$ in the

$$\tilde{k}_n(x,t) = \mathrm{X}(x) V \mathrm{N}(t);$$

$$\textbf{(23)}$$

$$n_i(t) = (2\mathrm{i}+1)\int_0^1 \tilde{k}(x,t) P_i(x)\, dx$$

$$\tilde{k}_{n,n}(x,t) = \mathrm{X}(x) V \mathrm{K} V^T \mathrm{X}^T(t);\ \tilde{k} = \left[\tilde{k}_{ij}\right]_{i,j=0}^{n}$$

$$\tilde{k}_{ij} = (2i+1)\int_0^1 n_i(t) P_j(t)\, dt \quad \textbf{(24)}$$

Furthermore, we get.

$$\tilde{k}_{n,n}(x,t)\,\tilde{u}_n(t) = X(x) V \tilde{K} \tilde{X}(T) V A;\ \tilde{X}(t) = X^T(t) X(t)$$

$$\textbf{(25)}$$

Substituting $\mathrm{K}\tilde{}_{n,n}(x,t) u\tilde{}_n(t)$ of (25) into (18), we get

$$\tilde{\tilde{\mathrm{X}}} = \int_0^1 \tilde{\mathrm{X}}(t)\, dt$$

(26)

$$\tilde{u}_n(x) = L(x) + \mathrm{X}(x) V \mathrm{K}^{\sim} V^T \tilde{\tilde{\mathrm{X}}} V \mathrm{A}$$

Substituting $\tilde{u}_n(x)$ in the left side of (1), $\tilde{k}_{n,n}(x,t)$ and $\tilde{u}_n(t)$ in the wright side, we get

$$\tilde{u}_n(x) = X(x) V \left(I_n - K V^T \tilde{\tilde{X}} V\right)^{-1} L \quad \textbf{(27)}$$

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Numerical Examples:

We developed MATLAB R2019b code based on the outlined method to solve two Fredholm integral equations of the second kind with weak singularities, which are relevant to the analysis of the flat stamp problem. Our computations focus on the stress distribution and intensity parameters at the surface of the functionally graded material (FGM) coating under the flat stamp load, and we compare our results with those from earlier studies. [3]

Example 1:

Think about a stamp that is flat and has $a/A=0.1, m=0.3$ Table 1 indicates the powers of stress singularity, β,α , In the leading order, respectively $x=-a$ and The rear positions $x=a$ ends Related to the stamp at a stiffness ratio Γ_3=8,2,1,1/2,1/8corresponding to different values [3] η=0.1,0.3,0.5

See Table 1: Stress intensity factors for the flat stamp, calculated using the interpolation method based on Barycentric Lagrange [3].

Γ_3	η =0.1 α=-0.5091 β=-0.4909		η =0.3 α=0.5272 β=-0.4728		η =0.5 α=-0.5452 β=-0.4528	
	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$
8	0.2765	0.2829	0.2695	0.2879	0.2610	0.2922
2	0.3019	0.3043	0.2987	0.3056	0.2945	0.305
2	0.3177	0.3178	0.3167	0.3167	0.3148	0.3147
6	0.3362	0.3337	0.3377	0.3301	0.3381	0.3258
1/8	0.3851	0.3765	0.3929	0.3669	0.3995	0.3568

See Table 2: Stress intensity factors for the flat stamp, obtained using the method derived from complex function theory [3]

Γ_3	η =0.1 α=-0.5091 β=-0.4909		η =0.3 α=0.5272 β=-0.4728		η =0.5 α=-0.5452 β=-0.4528	
	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$
8	0.2767	0.2831	0.2696	0.2882	0.2613	0.2924
2	0.3022	0.3046	0.2976	0.3058	0.2948	0.3058
2	0.3179	0.3180	0.3167	0.3169	0.3149	0.3149
6	0.3365	0.3339	0.3381	0.3303	0.3385	0.3260
1/8	0.3853	0.3767	0.3931	0.3671	0.3998	0.3569

See Table 3: Stress intensity factors for the flat stamp, Through the application of the First-kind Legendre polynomials in their shifted form for Example (1)

Γ_3	η =0.1 α=-0.5091 β=-0.4909		η =0.3 α=0.5272 β=-0.4728		η =0.5 α=-0.5452 β=-0.4528	
	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$
8	0.2769	0.2833	0.2696	0.2885	0.2615	0.2926
2	0.3025	0.3048	0.2991	0.3060	0.2949	0.3062
2	0.3182	0.3182	0.3171	0.3171	0.3151	0.3151
6	0.3366	0.3341	0.3382	0.3305	0.3386	0.3261
1/8	0.3855	0.3768	0.3933	0.3673	0.3999	0.3572

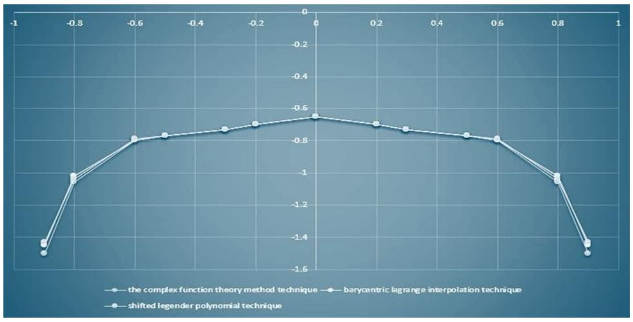

Fig 3. Surface stress distribution in the FGM coating under the influence of a flat stamp in exp (1)

Example 2:

Think about a stamp that is flat and has $a/A=0.5,\ m=0.3$ Table 2 Represents the powers associated with stress singularity, β,α, respectively at the leading $x=-a$ and the trailing $x=a$ ends of the stamp at a stiffness ratio $\Gamma_3=8,2,1,1/2,1/8$ corresponding For varying values [3] $\eta=0.1,0.3,0.5$

See Table 4: Stress intensity factors for the flat stamp, calculated using the interpolation method based on Barycentric Lagrange [3]

Γ_3	$\eta=0.1$ $\alpha=-0.5091$ $\beta=-0.4909$		$\eta=0.3$ $\alpha=0.5272$ $\beta=-0.4728$		$\eta=0.5$ $\alpha=-0.5452$ $\beta=-0.4528$	
	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$
8	0.1969	0.2764	0.1749	0.2418	0.1545	0.2631
2	0.2653	0.2736	0.2560	0.2809	0.2462	0.2872
2	0.3178	0.3178	0.3167	0.3167	0.3147	0.3148
6	0.3892	0.3798	0.3975	0.3692	0.4049	0.3585
1/8	0.6174	0.5839	0.6506	0.5508	0.6829	0.5181

See Table 5: Stress intensity factors for the flat stamp, obtained using the method derived from complex function theory [3]

Γ_3	$\eta=0.1$ $\alpha=-0.5091$ $\beta=-0.4909$		$\eta=0.3$ $\alpha=0.5272$ $\beta=-0.4728$		$\eta=0.5$ $\alpha=-0.5452$ $\beta=-0.4528$	
	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$
8	0.1973	0.2746	0.1754	0.2422	0.1549	0.2635
2	0.2657	0.2740	0.2565	0.2814	0.2467	0.2876
2	0.3182	0.3182	0.3171	0.3171	0.3151	0.3151
6	0.3895	0.3800	0.3979	0.3696	0.4053	0.3587
1/8	0.6178	0.5844	0.6510	0.5511	0.6834	0.5185

See Table 6: Stress intensity factors for the flat stamp, Through the application of the First-kind Legendre polynomials in their shifted form for Example (2)

Γ_3	$\eta=0.1$ $\alpha=-0.5091$ $\beta=-0.4909$		$\eta=0.3$ $\alpha=0.5272$ $\beta=-0.4728$		$\eta=0.5$ $\alpha=-0.5452$ $\beta=-0.4528$	
	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$	$k_1^{\sim}(-a)/La^{\alpha}$	$k_1^{\sim}(a)/La^{\beta}$
8	0.1971	0.2745	0.1752	0.2421	0.1547	0.2633
2	0.2655	0.2738	0.2563	0.2812	0.2465	0.2874
2	0.3180	0.3179	0.3169	0.3169	0.3149	0.3150
6	0.3894	0.3799	0.3978	0.3695	0.4052	0.3586
1\8	0.6176	0.5842	0.6508	0.5509	0.6832	0.5183

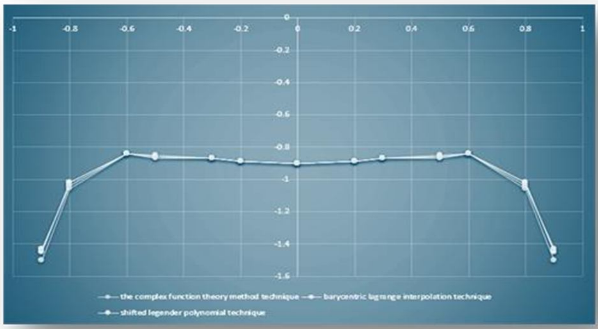

Fig4. Surface stress distribution in the FGM coating under the influence of a flat stamp in exp (2)

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