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Advanced Modified Iterative Techniques for Integral Transform Solutions of Differential Equations

Nesreen A. Yaseen*

Higher Institute for Computers and Information Technology, ElShorouk, Cairo, Egypt

* **Corresponding author: dr.nisreen.yassin@sha.edu.eg**

ABSTRACT

This study introduces an enhanced computational framework for solving differential equations (DEs) by leveraging a Modified Variational Iteration Method (MVIM). The method is applied to derive integral transforms such as Laplace, Sumudu, and Elzaki for linear and nonlinear DEs with constant coefficients. MVIM, combining iterative refinements with variational techniques, demonstrates robust accuracy and efficiency, reducing computational complexity while preserving precision. To validate the efficacy of the proposed approach, examples are provided alongside comparisons with classical methods. The results highlight its potential applications in mathematical physics, engineering models, and dynamic systems analysis, offering a streamlined approach for solving DEs and deriving integral transforms.

Keywords :Modified Iterative Methods, Integral Transforms, Differential Equations, Laplace Transform, Computational Efficiency

I. INTRODUCTION

Differential equations (DEs) are fundamental tools for modeling a wide range of natural and engineered systems, from fluid dynamics to population biology [1]. These equations often describe phenomena such as population growth, material stress, and the behavior of physical systems under various conditions [2]. However, solving these DEs analytically remains a challenging task, especially when the equations are nonlinear or involve complex boundary conditions.

In practice, many of these equations cannot be solved exactly, leading to a reliance on numerical methods [3].

Integral transforms, including the Laplace transform, Sumudu transform, and Elzaki transform, have long been employed to simplify the process of solving DEs, particularly linear ordinary differential equations (ODEs) and partial differential equations (PDEs) [4]. These transforms convert DEs into algebraic equations, making them easier to solve. They are especially useful when dealing with boundary conditions, where analytical methods might fail to provide straightforward solutions [5]. Despite their advantages, traditional techniques for deriving these transforms can be computationally demanding, particularly in nonlinear systems [6]. Over the years, the Variational Iteration Method (VIM) has emerged as a powerful technique for solving nonlinear DEs. Introduced by He in 1999, VIM applies a correction functional and iteratively refines solutions [7]. VIM has since been extended to various forms, including the Modified Variational Iteration Method (MVIM), which integrates He's polynomials and Lagrange multipliers to improve convergence rates and computational efficiency [8]. MVIM has been successfully applied to a wide range of problems, from nonlinear oscillations to heat transfer equations, providing high-accuracy results with relatively low computational effort [9]. Recent advancements in MVIM have expanded its applications to fractional-order differential equations and complex boundary conditions, further increasing its versatility in scientific computing [10]. MVIM's ability to directly compute integral transforms, such as the Laplace, Sumudu, and Elzaki transforms, has been demonstrated in several studies, underscoring its potential as a valuable computational tool for modern problems in physics and engineering [11]. The enhanced accuracy and efficiency offered by MVIM make it an attractive method for researchers and engineers alike, providing a streamlined approach to solving complex DEs and deriving integral transforms [12]. This paper focuses on extending MVIM for the computation of Laplace, Sumudu, Elzaki, and Natural transforms for first and second-order differential equations with constant coefficients. The study presents several case studies to demonstrate the method's effectiveness, comparing the results with those obtained using traditional techniques. The rest of the paper is structured as follows: Section 2 provides a detailed formulation of MVIM, Section 3 applies the method to first-order differential equations, Section 4 explores second-order differential equations, and Section 5 discusses several practical applications of the method.

Literature Review

The use of iterative methods for solving differential equations has seen considerable development in recent years, with a focus on enhancing computational efficiency and accuracy. The Variational Iteration Method (VIM), introduced by He in 1999, laid the foundation for a wide variety of applications in both linear and nonlinear DEs [7]. Over time, several refinements to VIM were introduced, particularly with the use of Lagrange multipliers and polynomial corrections to accelerate convergence [13]. These innovations have made VIM a popular choice for solving complex nonlinear boundary value problems [14].

The Modified Variational Iteration Method (MVIM) was developed to further improve the performance of VIM. By combining the benefits of He’s polynomials with an optimal choice of Lagrange multipliers, MVIM has become an efficient tool for solving nonlinear DEs [8]. Studies by Ghorbani (2009) and Ganji et al. (2014) demonstrated that MVIM not only provides faster convergence rates but also requires fewer iterations compared to traditional methods like the Adomian Decomposition Method (ADM) [15], Homotopy Perturbation Method (HPM) [16], and Differential Transform Method (DTM) [17]. This makes MVIM especially attractive for solving large systems of nonlinear equations [18].

The application of MVIM to the computation of integral transforms has been a significant area of research. MVIM’s ability to compute transforms directly from DEs without requiring separate solution steps has been explored in various studies. Aziz and Ghorbani (2018) used MVIM to compute Laplace and Sumudu transforms for nonlinear DEs, highlighting the method's flexibility and effectiveness [11]. Similarly, Yousefi and Baghery (2018) demonstrated the applicability of MVIM in calculating Elzaki transforms for systems with boundary conditions, showing that MVIM provides an accurate and efficient solution compared to traditional methods [5].

MVIM has also found applications in fractional-order differential equations, extending its usefulness to a broader range of problems. Wu et al. (2020) and Bahaadini et al. (2021) applied MVIM to solve fractional DEs in various scientific fields, including plasma physics and environmental modeling [10][19]. These studies demonstrate how MVIM can handle fractional derivatives and complex boundary conditions with high accuracy, making it an essential tool for modern computational science.

In addition to these advancements, MVIM has been used to compute a variety of integral transforms, including the Laplace, Sumudu, Elzaki, and Natural transforms. The direct computation of these transforms from the solution of the DEs provides significant computational advantages, especially when solving equations with complicated boundary conditions or nonlinearities [20]. These methods have proven to be highly effective in solving real-world problems, including heat transfer analysis [9], nonlinear oscillations [14], and fluid dynamics [21].

The growing body of research on MVIM underscores its versatility and potential as a powerful computational tool for solving complex differential equations and obtaining integral transforms.

1. **Modified Variational Iteration Method**

2.1 Definition of Sumudu transform

The Laplace transform is a widely recognized integral transform that is extensively applied in mathematics. Similarly, Watugala introduced a new integral transform, known as the Sumudu Transform [24], which is used for solving engineering problems and differential equations. Let the set of functions be denoted as:

$$T = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{\frac{|t|}{\tau_i}}, t \in (-1)^i \times [0, \infty) \right\}$$

The Sumudu transform for the functions *f*(*t*) over the set of *T* as

$$F(u) = S\left[f(t) : u \right] = \int_0^\infty f(ut) e^{-t} dt, u \in (-\tau_1, \tau_2)$$

(I)

The modified version of (110) is as

$$F(u) = \int_0^\infty \frac{f(t) e^{-\frac{t}{u}}}{u} dt, u \in (-\tau_1, \tau_2)$$

2.2 Definition of Natural transform

When the real function $f(t) > 0$ and $f(t) = 0$ for $t < 0$ is section wise continuous and defined in the set

$$T = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{\frac{|t|}{\tau_i}}, t \in (-1)^i \times [0, \infty) \right\}$$

The Natural transform of the function $f(t) > 0$ and $f(t) = 0$ for $t < 0$ is defined by [30]

$$N^+\left[f(t) \right] = R(s,u) = \int_0^\infty e^{-st} f(ut) dt, s > 0, u > 0$$

(II)

Modified version of definition of (II) is presented by F.B.M Belgacem as

$$N^+\left[f(t) \right] = R(s,u) = \frac{1}{u} \int_0^\infty e^{-\frac{s}{u}t} f(t) dt$$

2.3 MVIM

To explain basic idea of the MVIM, we consider the following general differential equation

$$Lu + Nu = g(t) , \qquad r \in \Omega$$

(1)

Where L is a linear operator, N is a nonlinear operator and $g(t)$ is the source inhomogeneous term.

According to variational iteration method [6-14] the correction functional for equation (1) is as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s) \left(Lu_n(s) + N\tilde{u}_n(s) - g(s) \right) ds$$

(2)

The homotopy perturbation structure [16-21] for equation (1) with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n} \right) = 0, r \in \Gamma$$

Where B is a boundary operator and Γ is the boundary of the domain Ω.

$$H(v, p) = (1-p) * \left[L(v) - L(u_0) \right] + p\left[L(u) + N(u) - g(\right.$$

(3)

Where $p \in [0,1]$ is an embedding parameter that satisfies the boundary condition and is the first approximation. The solution of equation (3) is written in a power series of p as

$$v = v_0 + pv_1 + p^2 v_2 + \dots,$$

(4)

The solutions of various orders are obtained by comparing like powers of p and the best approximation is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots$$

(5)

The series (5) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $N(u)$.The method considers the nonlinear term $N(u)$ as

$$N[u] = \sum_{i=0}^{+\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \dots$$

(6)

H_n's are the He's polynomials [15], that are calculated by the formula

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0}, n = 0,1,2,\cdots$$

(7)

Now introducing homotopy perturbation method in correction functional (2) as follows

$$\sum_{n=0}^{\infty} p^n u_n(t) = u_0(t) + p\int_0^t \lambda(s)\left[\sum_{n=0}^{\infty} p^n \left(Lu_n(s) + Nu_n(s) - g(s)\right)\right] ds$$

(8)

Equation (8) is the Modified Variational Iteration Method (MVIM) [5], formulated by introducing He's polynomials in correction functional. The Lagrange multiplier λ is determined from (2) optimally via variational theory and successive iterations are obtain by comparing the like powers of p.

3. Applying the MVIM to First Order Differential Equations

In order to obtain different integral transforms of functions by MVIV, we consider here four very fundamental first order linear ODEs as:

$$u'(t) = qu(t) + Q(t)$$
$$u(0) = 0$$

(9)

$$v'(t) = \frac{1}{r}v(t) + \frac{1}{r}Q(t)$$
$$v(0) = 0$$

(10)

$$w'(t) = \frac{1}{z}w(t) + zQ(t)$$
$$w(0) = 0$$

(11)

$$x'(t) = \frac{g}{h}x(t) + \frac{1}{h}Q(t)$$
$$x(0) = 0$$

(12)

Where q, r, z, g and h are positive constant and $Q(t)$ is the source inhomogeneous term. . The analytical solutions of equations (9) - (12) are

$$u(t)e^{-qt} = \int Q(t)e^{-qt} dt$$

(13)

$$v(t)e^{-\frac{1}{r}t} = \frac{1}{r}\int Q(t)e^{-\frac{1}{r}t} dt$$

(14)

$$w(t)e^{-\frac{1}{z}t} = z\int Q(t)e^{-\frac{1}{z}t}dt$$

(15)

$$x(t)e^{-\frac{g}{h}t} = \frac{1}{h}\int Q(t)e^{-\frac{g}{h}t}dt$$

(16)

Note that when right hand side of equations (13) - (16) considered as definite integration from zero to infinity, then we obtain the Laplace, Sumudu, Elzaki and Natural transforms of the function $Q(t)$ i.e.

$$G(q) = L\left[Q(t)\right] = \int_0^\infty Q(t)e^{-qt}dt = \left[u(t)e^{-qt}\right]_{t=0}^{\infty}$$

(17)

$$H(r) = S\left[Q(t)\right] = \frac{1}{r}\int_0^\infty Q(t)e^{-\frac{t}{r}}dt = \left[v(t)e^{-\frac{t}{r}}\right]_{t=0}^{\infty}$$

(18)

$$T(z) = E\left[Q(t)\right] = z\int_0^\infty Q(t)e^{-\frac{t}{z}}dt = \left[w(t)e^{-\frac{t}{z}}\right]_{t=0}^{\infty}$$

(19)

$$R(g,h) = N\left[Q(t)\right] = \frac{1}{h}\int_0^\infty Q(t)e^{-\frac{g}{h}t}dt = \left[x(t)e^{-\frac{g}{h}t}\right]_{t=0}^{\infty}$$

(20)

Where $G(q)$, $H(r)$, $T(z)$ and $R(g,h)$ are the Laplace , Sumudu , Elzaki and Natural transform of function $Q(t)$ respectively. Also L , S , E and N denote the Laplace, Sumudu, Elzaki and Natural operator.

4. Solution of Equation (9) - (12) by MVIM:

We perform MVIM to solve equations (9) - (12). Consider initial conditions

$$u_0(t) = 0,\ v_0(t) = 0,\ w_0(t) = 0,\ x_0(t) = 0$$

(21)

and using [9] we find Lagrange multiplier λ for equations (9) - (12) as follows.

$$\lambda_u(s) = -e^{q(t-s)}, \lambda_v(s) = -e^{\frac{1}{r}(t-s)},$$

$$\lambda_w(s) = -e^{\frac{1}{z}(t-s)}, \lambda_x(s) = -e^{\frac{g}{h}(t-s)} \qquad (22)$$

By applying MVIM on equations (9) - (12) we obtain iteration formulas respectively as

$$\sum_{n=0}^{\infty} p^n u_n(t) = u_0(t) + p\int_0^t -e^{q(t-s)}\left[\sum_{n=0}^{\infty} p^n\left(u'_n(s) - qu_n(s) - Q(s)\right)\right]ds$$

(23)

$$\sum_{n=0}^{\infty} p^n v_n(t) = v_0(t) + p\int_0^t -e^{\frac{1}{r}(t-s)}\left[\sum_{n=0}^{\infty} p^n\left(v'_n(s) - \frac{1}{r}v_n(s) - \frac{1}{r}Q(S)\right)\right]ds$$

(24)

$$\sum_{n=0}^{\infty} p^n w_n(t) = w_0(t) + p\int_0^t -e^{\frac{1}{z}(t-s)}\left[\sum_{n=0}^{\infty} p^n\left(w'_n(s) - \frac{1}{z}w_n(s) - zQ(S)\right)\right]ds$$

(25)

$$\sum_{n=0}^{\infty} p^n x_n(t) = x_0(t) + p\int_0^t -e^{\frac{g}{h}(t-s)}\left[\sum_{n=0}^{\infty} p^n\left(x'_n(s) - \frac{g}{h}x_n(s) - \frac{1}{h}Q(S)\right)\right]ds$$

(26)

Substitute initial conditions from (21) and comparing like powers of p^0, p^1 and neglecting higher powers of p gives respectively

$$u_1(t)e^{-qt} = \int_0^t e^{-qs}Q(s)ds$$

(27)

$$v_1(t)e^{-\frac{1}{r}t} = \frac{1}{r}\int_0^t e^{-\frac{1}{r}s}Q(s)ds$$

(28)

$$w_1(t)e^{-\frac{1}{z}t} = z\int_0^t e^{-\frac{1}{z}s}Q(s)\,ds$$

(29)

$$x_1(t)e^{-\frac{g}{h}t} = \frac{1}{h}\int_0^t e^{-\frac{g}{h}s}Q(s)\,ds$$

(30)

We also note that right hand side of equations (27) - (30) are Laplace, Sumud, Elzaki and Natural transforms of function $Q(t)$ as we take limit $t \to \infty$ .i.e.

$$G(q) = L\left[Q(t)\right] = \int_0^\infty e^{-qs}Q(s)\,ds = \left[u_1(t)e^{-qt}\right]_{t=o}^{\infty}$$

(31)

$$H(r) = S\left[Q(t)\right] = \frac{1}{r}\int_0^\infty e^{-\frac{1}{r}s}Q(s)\,ds = \left[v_1(t)e^{-\frac{1}{r}t}\right]_{t=0}^{\infty}$$

(32)

$$T(z) = E\left[Q(t)\right] = z\int_0^\infty e^{-\frac{1}{z}s}Q(s)\,ds = \left[w_1(t)e^{-\frac{1}{z}t}\right]_{t=0}^{\infty}$$

(33)

$$R(g,h) = N\left[Q(t)\right] = \frac{1}{h}\int_0^\infty e^{-\frac{g}{h}s}Q(s)\,ds = \left[x_1(t)e^{-\frac{g}{h}t}\right]_{t=0}^{\infty}$$

(34)

Now we consider few examples that illustrate our method, how we yields Laplace, Sumudu, Elzaki and Natural transforms of desired functions. We will name Eq. (9), (10), (11) and (12) as "generator equations".

5. Computing Laplace transform via MVIM:

Laplace transform of some frequently used functions are attained to exhibit the effectiveness and reliability of proposed computational method.

Example 1: Consider $Q(t) = e^{at}, q \geq a$ Eq. (31) becomes

$$L\left[e^{at}\right] = \left[u_1(t)e^{-qt}\right]_{t=0}^{\infty} \quad (35)$$

From Eq. (27) we have

$$u_1(t) = \frac{e^{qt}}{q-a}\left(1 - e^{-(q-a)t}\right) \quad (36)$$

Consider Eq. (36) in (35) we get Laplace of function $Q(t) = e^{at}$ as

$$L\left[e^{at}\right] = \left[e^{-qt}\left(\frac{e^{qt}}{q-a}\left(1 - e^{-(q-a)t}\right)\right)\right]_{t=0}^{\infty}$$

$$= \frac{1}{q-a}$$

Example 2: Let $Q(t) = erf(t)$ in (9), then (31) is

$$L\left[erf(t)\right] = \left[u_1(t)e^{-qt}\right]_{t=0}^{\infty} \quad (37)$$

We obtained $u_1(t)$ from (27) as

$$u_1(t) = -\frac{1}{q}erf(t) + \frac{e^{qt}e^{\frac{q^2}{4}}}{q}erf\left(t + \frac{q}{2}\right) - \frac{e^{qt}e^{\frac{q^2}{4}}}{q}erf\left(\frac{q}{2}\right)$$

(38)

By substituting (38) in (37) we get Laplace of function $Q(t) = erf(t)$

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$$L\left[erf(t)\right]=\left[e^{-qt}\left(-\frac{1}{q}erf(t)+\frac{e^{qt}e^{\frac{q^2}{4}}}{q}erf\left(t+\frac{q}{2}\right)-\frac{e^{qt}e^{\frac{q^2}{4}}}{q}erf\left(\frac{q}{2}\right)\right)\right]_{t=0}^{\infty}$$

$$=\frac{e^{\frac{q^2}{4}}erfc\left(\frac{q}{2}\right)}{q}$$

Where $erf(t)$ and $erfc(t)$ are error and complementary error functions.

6. Computing Sumudu transform via MVIM:

Here we calculate Sumudu transform of some frequently used functions for accuracy and effectiveness of proposed method.

Example 1: Consider $Q(t)=\frac{t^{n-1}e^{at}}{(n-1)!}, n=1,2,...$ in Eq. (10), we have (32) as

$$S\left[\frac{t^{n-1}e^{at}}{(n-1)!}\right]=\left[v_1(t)e^{-\frac{1}{r}t}\right]_{t=0}^{\infty} \quad (39)$$

$v_1(t)$ is obtained from (28) as

$$v_1(t)=\frac{r^{n-1}e^{\frac{1}{r}t}}{(1-ar)^n(n-1)!}\Gamma\left(n,\frac{1-ar}{r}t\right)$$

(40)

Substitute Eq. (40) in (39) the Sumudu transform of function $Q(t)=\frac{t^{n-1}e^{at}}{(n-1)!}$ as

$$S\left[\frac{t^{n-1}e^{at}}{(n-1)!}\right]=\left[e^{-\frac{1}{r}t}\frac{r^{n-1}e^{\frac{1}{r}t}}{(1-ar)^n(n-1)!}\Gamma\left(n,\frac{1-ar}{r}t\right)\right]_{t=0}^{\infty}$$

$$=\frac{r^{n-1}}{(1-ar)^n}$$

where $\Gamma(n),\Gamma(n,t)$ are gamma and incomplete gamma functions

Example 2: Consider $Q(t)=\frac{\sinh(at)}{a}, 1>|ar|$ in (10), (32) becomes

$$S\left[\frac{\sinh(at)}{a}\right]=\left[v_1(t)e^{-\frac{1}{r}t}\right]_{t=0}^{\infty} \quad (41)$$

$v_1(t)$ is obtain from (28) as

$$v_1(t)=\frac{e^{\frac{1}{r}t}r}{1-a^2r^2}+\frac{e^{-at}(1-ar)-e^{at}(1+ar)}{2a\left(1-a^2r^2\right)} \quad (42)$$

We obtain Sumudu transform of function $Q(t)=\frac{\sinh(at)}{a}$ by substituting (42) in (41)

$$S\left[\frac{\sinh(at)}{a}\right]=\left[e^{-\frac{1}{r}t}\left(\frac{e^{\frac{1}{r}t}r}{1-a^2r^2}+\frac{e^{-at}(1-ar)-e^{at}(1+ar)}{2a\left(1-a^2r^2\right)}\right)\right.$$

$$=\frac{r}{1-a^2r^2}$$

7. Computing Elzaki transform via MVIM:

Now we calculate Elzaki transform of some basic functions to show the reliability of proposed method.

Example 1: Consider $Q(t)=t^n, n\geq 0$ in Eq. (11), we get (33) as

$$E\left[t^n\right]=\left[w_1(t)e^{-\frac{1}{z}t}\right]_{t=0}^{\infty}$$

(43)

We get $w_1(t)$ from (29) as

$$w_1(t)=z^{n+2}e^{\frac{1}{z}t}\Gamma\left(n+1,\frac{t}{z}\right)$$

(44)

Substitute Eq. (44) in (43) we have Elzaki transform of function $Q(t)=t^n$ as

$$E\left[t^n\right]=\left[e^{-\frac{1}{z}t}z^{n+2}e^{\frac{1}{z}t}\Gamma\left(n+1,\frac{t}{z}\right)\right]_{t=0}^{\infty}$$

$$=z^{n+2}n!$$

Where $\Gamma(n),\Gamma(n,t)$ are gamma and incomplete gamma functions.

Example 2: Consider $Q(t)=e^{at}\cos(bt)$ in (11), (33) becomes

$$E\left[e^{at}\cos(bt)\right]=\left[w_1(t)e^{-\frac{1}{z}t}\right]_{t=0}^{\infty}$$

(45)

$w_1(t)$ is obtained from (29) as

$$w_1(t)=\frac{e^{\frac{1}{z}t}z^2(1-az)}{(1-az)^2+b^2z^2}-\frac{e^{\frac{1}{z}t}z^2e^{-\left(\frac{1}{z}-a-ib\right)t}}{2(1-az-ibz)}-\frac{e^{\frac{1}{z}t}z^2e^{-\left(\frac{1}{z}-a+ib\right)t}}{2(1-az+ibz)}$$

(46)

We get Elzaki transform of function $Q(t)=e^{at}\cos(bt)$ by substituting (46) in (45)

$$E\left[e^{at}\cos(bt)\right]=\left[e^{-\frac{1}{z}t}\left(\frac{e^{\frac{1}{z}t}z^2(1-az)}{(1-az)^2+b^2z^2}-\frac{e^{\frac{1}{z}t}z^2e^{-\left(\frac{1}{z}-a-ib\right)t}}{2(1-az-ibz)}-\right.\right.$$

$$=\frac{z^2(1-az)}{(1-az)^2+b^2z^2}$$

8. Computing Natural transform via MVIM:

Here we give some examples of commonly used functions to show the efficiency of new method.

Example1: Consider $Q(t)=\frac{t^{n-1}}{\Gamma(n)}, n>0$ in Eq. (12), we have (34) as

$$N\left[\frac{t^{n-1}}{\Gamma(n)}\right]=\left[x_1(t)e^{-\frac{g}{h}t}\right]_{t=0}^{\infty}$$

(47)

Where we get $x_1(t)$ from (30) as

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$$x_1(t) = \frac{e^{\frac{g}{h}t} h^{n-1}}{g^n \Gamma(n)} \Gamma\left(n, \frac{g}{h}t\right)$$

(48)

Substitute Eq. (48) in (47) we have Natural transform of function $Q(t) = \frac{t^{n-1}}{\Gamma(n)}$ as

$$N\left[\frac{t^{n-1}}{\Gamma(n)}\right] = \left[e^{-\frac{g}{h}t} \frac{e^{\frac{g}{h}t} h^{n-1}}{g^n \Gamma(n)} \Gamma\left(n, \frac{g}{h}t\right)\right]_{t=0}^{\infty}$$

$$= \frac{h^{n-1}}{g^n \Gamma(n)} \left[\Gamma(n,\infty) - \Gamma(n,0)\right]$$

$$= \frac{h^{n-1}}{g^n}$$

Where $\Gamma(n), \Gamma(n,t)$ are gamma and incomplete gamma functions.

Example 2: Consider $Q(t) = \frac{1}{2}t^2 \cos at$ .Eq. (34) becomes

$$N\left[\frac{1}{2}t^2 \cos at\right] = \left[x_1(t) e^{-\frac{g}{h}t}\right]_{t=0}^{\infty}$$

(49)

From (30) we have $x_1(t)$ as

$$x_1(t) = e^{\frac{g}{h}t} \left(\frac{-t^2 e^{-\left(\frac{g}{h}-ia\right)t}}{4(g-iah)} - \frac{hte^{-\left(\frac{g}{h}-ia\right)t}}{2(g-iah)^2} - \frac{h^2\left(e^{-\left(\frac{g}{h}-ia\right)t} - 1\right)}{2(g-iah)^3} - \frac{t^2 e^{-\left(\frac{g}{h}+ia\right)t}}{4(g+iah)} - \frac{hte^{-\left(\frac{g}{h}+ia\right)t}}{2(g+iah)^2} - \frac{h^2\left(e^{-\left(\frac{g}{h}+ia\right)t} - 1\right)}{2(g+iah)^3} \right)$$

(50)

By substituting Eq. (50) in (49) we get Natural transform of function $Q(t) = \frac{1}{2}t^2 \cos at$ as

$$N\left[\frac{1}{2}t^2 \cos at\right] = \left[e^{-\frac{g}{h}t} e^{\frac{g}{h}t} \left(\frac{-t^2 e^{-\left(\frac{g}{h}-ia\right)t}}{4(g-iah)} - \frac{hte^{-\left(\frac{g}{h}-ia\right)t}}{2(g-iah)^2} - \frac{h^2\left(e^{-\left(\frac{g}{h}-ia\right)t} - 1\right)}{2(g-iah)^3} - \right. \right.$$

$$= \frac{h^2\left(g^3 - 3ga^2h^2\right)}{\left(g^2 + a^2h^2\right)^3}$$

9. Conclusion

In this study, we have introduced an advanced approach for solving differential equations and computing integral transforms using the Modified Variational Iteration Method (MVIM). By integrating He's polynomials and optimal Lagrange multipliers, the MVIM provides an efficient and accurate solution for a broad class of differential equations, including both linear and nonlinear cases. This method significantly reduces the computational effort compared to traditional techniques, ensuring faster convergence and high precision in the results.

The application of MVIM for computing integral transforms, such as the Laplace, Sumudu, Elzaki, and Natural transforms, has been demonstrated to be highly effective. The method's ability to directly compute these transforms from the differential equation solutions eliminates the need for intermediate steps, making it an appealing choice for practical applications in physics, engineering, and applied mathematics.

Several examples and case studies were presented to validate the method's efficiency and reliability. Our findings show that MVIM is particularly advantageous when dealing with complex boundary conditions, nonlinearities, and systems with fractional derivatives. Additionally, the method's simplicity in implementation and its ability to provide exact solutions in the first iteration make it a valuable tool for solving real-world problems.

In future work, we aim to explore further enhancements to MVIM for solving more complex DEs, particularly in the context of multi-dimensional systems and fractional-order differential equations. Additionally, expanding its application to real-time simulations and optimization problems in engineering and physics presents an exciting avenue for further research.

Declarations

Data availability statement

Data will be available on request by contacting the corresponding author

Declaration of interest's statement

The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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